$$h = \frac{\lambda}{2} g(x), \quad g(x) \equiv \int_{-1}^{x} \int_{0}^{\eta} \int_{-1}^{1} K(t, \xi)(t^2 - 1)\, dt d\xi d\eta$$

Using the expression obtained for $h$ and the isoperimetric condition, we find that

$$\lambda = \left[\int_0^1 g(x)\, dx\right]^{-1} \approx 2.269$$

Figure 2 shows the optimal thickness distribution (because of symmetry, only the region $x > 0$ is shown).

We shall now show that when $D$ is linearly dependent on $h$ $(\alpha \to \infty)$, the necessary condition of optimality also becomes sufficient. In fact, let $h^*$ and $U^*$ be solutions satisfying the condition of optimality, $h$ be an arbitrary thickness distribution and $U$ be the corresponding deflection function. Then

$$\Delta\lambda = \lambda(h^*) - \lambda(h) = \frac{I_1(h^*, U^*)}{I_2(U^*)} - \frac{I_1(h, U)}{I_2(U)}$$

We shall show that $\Delta\lambda \geqslant 0$. To do this, we take into account the condition of optimality (3.3), the isoperimetric condition (3.1) and the properties of the functions $h^*$, $U^*$ and $h$, $U$, to arrive at the following estimates:

$$\Delta\lambda = \frac{I_1(h^*, U^*)}{I_2(U^*)} - \frac{I_1(h, U)}{I_2(U)} \geqslant \frac{I_1(h^*, U^*)}{I_1(U^*)} - \frac{I_1(h, U^*)}{I_2(U^*)} =$$
$$\frac{1}{I_2(U^*)} \iint_G \left\{\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 U}{\partial x^2}\frac{\partial^2 U}{\partial y^2} - \left(\frac{\partial^2 U}{\partial x \partial y}\right)^2\right]\right\}(h^* - h)\, dxdy = \frac{1}{I_2(U^*)} \iint_G c^2 (h^* - h)\, dxdy = 0$$

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Translated by L. K.

UDC 539.3

# STRESS FUNCTIONS AND SOME A PRIORI ESTIMATES IN PLATE BENDING THEORY

PMM Vol. 40, № 3, 1976, pp. 528-535

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(Received December 12, 1974)

A system of equations of the Reissner plate bending theory is formulated in terms of stress functions. An estimate of the elastic energy is deduced from the variational principle for the stress functions. By using this estimate it is proved that

the solution of the system of Reissner equations tends to the solution of the Kirchhoff equation in terms of the energy norm as the plate thickness diminishes.

**1. Kirchhoff and Reissner models.** In a Cartesian coordinate system $x^\alpha$ (the Greek superscripts take on the values 1, 2), let us consider a linearly elastic anisotropic plate of constant thickness $h$, whose middle plane occupies a bounded simply-connected domain $\Omega$ with the smooth boundary $\Gamma$ in the $x^\alpha$ plane. The state of stress of the plate under bending is described by the tensor of the bending moments $m^{\alpha\beta}$ and the vector of the transverse forces $q^\alpha$. In the Reissner model $q^\alpha$ and $m^{\alpha\beta}$ are determined from the solution of the following variational problem: find the minimum of the functional

$$E = \int_\Omega \Phi(m^{\alpha\beta}, q^\beta)\, dx^1 dx^2 \tag{1.1}$$

among all the functions $m^{\alpha\beta}$ and $q^\alpha$ satisfying the constraints

$$m^{\alpha\beta}{}_{,\beta} - q^\alpha = 0, \quad q^\alpha{}_{,\alpha} = 0, \quad m^{\alpha\beta} = m^{\beta\alpha} \tag{1.2}$$

$$m^{\alpha\beta}\nu_\beta = M^\alpha(s), \quad q^\alpha\nu_\alpha = Q(s) \quad \text{on } \Gamma \tag{1.3}$$

Here the comma in the subscripts denotes differentiation with respect to $x^\alpha$, $\nu^\alpha$ is the vector of the external unit normal to the contour $\Gamma$, $s$ is the arclength along $\Gamma$, the external forces are considered applied only to the contour $\Gamma$ and reduce to the bending moment $M^\alpha(s)$ and the transverse force $Q(s)$, the elastic energy density is a quadratic form in $m^{\alpha\beta}$ and $q^\alpha$ of the form

$$2\Phi = A^{\alpha\beta\gamma\delta} m_{\alpha\beta} m_{\gamma\delta} + h^2 B^{\alpha\beta} q_\alpha q_\beta \tag{1.4}$$

The elastic compliance tensors $A^{\alpha\beta\gamma\delta}$ and $B^{\alpha\beta}$ are considered independent of the parameter $h$ (the factor $h^{-3}$ which is not essential later is omitted). The tensor $A^{\alpha\beta\gamma\delta}$ has the symmetry of the elastic models tensor.

Reissner obtained the variational problem (1.1) – (1.4) from the Castigliano variational principle on the basis of hypotheses relative to the stress tensor components [1]. Appropriate hypotheses relative to the displacement vector components and a derivation of the Reissner equations from the Lagrange variational principle are presented in [2]. The variational problem in displacements is dual [3] to the variational problem (1.1)–(1.4). The asymptotic accuracy of the Reissner model has been proved in [4].

In the Kirchhoff model $m^{\alpha\beta}$ and $q^\alpha$ are determined from the problem of seeking the minimum of the functional

$$E_K = \int_\Omega \Phi_K(m^{\alpha\beta})\, dx^1 dx^2 \tag{1.5}$$

among all functions $m^{\alpha\beta}$ and $q^\alpha$ which satisfy Eqs. (1.2) in the domain $\Omega$ and the following conditions on $\Gamma$

$$m^{\alpha\beta}\nu_\alpha\nu_\beta = M(s), \quad q^\alpha\nu_\alpha + \frac{d}{ds} m^{\alpha\beta}\tau_\alpha\nu_\beta = N(s) \tag{1.6}$$

Here $\tau^\alpha$ is a vector tangent to $\Gamma$ (the direction of traversal for which the domain $\Omega$ remains on the left is taken as positive), $\Phi_K = \Phi(m^{\alpha\beta}, 0)$, and the Kirchhoff forces $M$ and $N$ are related to $M^\alpha$ and $Q$ by the formulas

$$M = M^\alpha\nu_\alpha, \quad N = Q + \frac{d}{ds} M^\alpha\tau_\alpha$$

The quantity $M^\alpha\tau_\alpha$ is later considered a continuously differentiable function on $\Gamma$.

Notes. 1°. At first glance the variational formulations presented above seem to be astonishing since the natural space in which the minima of $E$ and $E_K$ should be sought is $L_2$, while the differential relations (1.2) and constraints on the set of measure zero (1.3) and (1.6) are imposed on the required functions. However, the constraints (1.2), (1.3) which are meaningful only for differentiable continuous functions admit of an extension to $L_2$ if they are written in the form [3]

$$\int_\Omega [m^{\alpha\beta}\psi_{(\alpha,\beta)} + q^\alpha (u_{,\alpha} + \psi_\alpha)]\, dx^1 dx^2 = \int_\Gamma (M^\alpha \psi_\alpha + Qu)\, ds \tag{1.7}$$

Here the parentheses in the subscripts denote the symmetrization operation, $\psi_\alpha$ and $u$ are arbitrary functions from $H_2^1(\Omega)$. (It is taken into account in (1.7) that the functional $E$ is independent of the antisymmetric part of the bending moments tensor, hence, the last relationship in (1.2) can be discarded by substituting the tensor $m^{(\alpha\beta)}$) instead of $m^{\alpha\beta}$ on the first equation in (1.2).

The constraints (1.2), (1.6) are extended to $L_2$ as follows ($u$ is any function from $H_2^2(\Omega)$):

$$-\int_\Omega m^{\alpha\beta} u_{,\alpha\beta}\, dx^1 dx^2 = \int_\Gamma \left(-M \frac{\partial u}{\partial \nu} + Nu\right) ds \tag{1.8}$$

Here $\partial/\partial\nu = \nu_\alpha \partial/\partial x^\alpha$. Let us note that the relationship (1.8) follows formally from (1.7), if we set $\psi_\alpha = -u_{,\alpha}$ in (1.7) and the integral over $\Gamma$ is converted by using the identities

$$M^\alpha = \nu^\alpha (M^\beta \nu_\beta) + \tau^\alpha (M^\beta \tau_\beta) \tag{1.9}$$

$$\int_\Gamma M^\alpha u_{,\alpha} ds = \int_\Gamma \left(M \frac{\partial u}{\partial \nu} - u \frac{d}{ds} M^\alpha \tau_\alpha\right) ds$$

2°. The functions $M^\alpha$ and $Q$ given on $\Gamma$ in the Reissner model should satisfy the conditions

$$\int_\Gamma Q\, ds = 0, \quad \int_\Gamma (M^\alpha - x^\alpha Q)\, ds = 0 \tag{1.10}$$

These conditions are obtained from (1.7) if we set

$$\psi_\alpha = 0,\ u = \text{const}; \quad \psi_\alpha = c_\alpha = \text{const}, \quad u = -c_\alpha x^\alpha$$

The quantities $M$ and $N$ in the Kirchhoff model should satisfy the contraints

$$\int_\Gamma N\, ds = 0, \quad \int_\Gamma (\nu^\alpha M - x^\alpha N)\, ds = 0 \tag{1.11}$$

The equalities (1.11) follow from (1.8) for $u = \text{const}$ and $u = c_\alpha x^\alpha$. The equivalence of the constraints (1.10) and (1.11) is easily proved by using the identities (1.9).

**2. Stress functions (*).** General solution of the equilibrium equations. Any solution of the equilibrium equations (1.2) can be represented locally as

$$m^{\alpha\beta} = e^{\beta\gamma}\chi^\alpha_{,\gamma} + e^{\alpha\beta}\chi, \quad q^\alpha = e^{\alpha\beta}\chi_{,\beta}, \quad \chi = {}^1/_2 \chi^\alpha_{,\alpha} \tag{2.1}$$

where $e^{\alpha\beta}$ are the Levi-Civita symbols ($e^{11} = e^{22} = 0,\ e^{12} = -e^{21} = 1$) and $\chi^\alpha$ are certain functions.

Actually, the general solution of the second equation of (1.2) locally has the form

$$q^\alpha = e^{\alpha\beta}\chi_{,\beta} \tag{2.2}$$

*) Assertions 1° and 6° are particular cases of the corresponding Reissner results for shells [5].

where $\chi$ is an arbitrary function. Rewriting the first equation in (1.2) as

$$(m^{\alpha\beta} - e^{\alpha\beta}\chi)_{,\beta} = 0$$

we obtain analogously to (2.2)

$$m^{\alpha\beta} - e^{\alpha\beta}\chi = e^{\beta\gamma}\chi^{\alpha}_{,\gamma} \tag{2.3}$$

where $\chi^\alpha$ are arbitrary functions. There remains to satisfy the symmetry condition for the bending moments tensor. Displacing (2.3) with $e_{\alpha\beta}$ we arrive at the third equation in (2.1).

It is easy to see that $\chi^\alpha$ are components of a pseudovector. Henceforth, we shall call $\chi^\alpha$ the stress functions.

2°. Arbitrariness in the selection of the stress functions. In order for the functions $\chi^\alpha$ and $\chi'^\alpha$ to correspond to the same state of stress, it is necessary and sufficient that ($c$, $c^\alpha$ are arbitrary constants)

$$\chi'^\alpha = \chi^\alpha + cx^\alpha + c^\alpha$$

Henceforth, we shall eliminate arbitrariness in the selection of the functions $\chi^\alpha$ by imposing the condition

$$\int_\Gamma \chi^\alpha ds = 0, \quad \int_\Gamma \chi ds = 0 \tag{2.4}$$

3°. Boundary conditions. Equations (1.2) are common to the Kirchhoff and Reissner models, hence, admissible states of stress can be represented in terms of stress functions in both the Kirchhoff and Reissner models. The difference will be in the boundary conditions for the stress functions.

According to (1.3) and (2.1), the stress functions in the Reissner model should satisfy the following relationships on $\Gamma$

$$d\chi^\alpha/ds - \tau^\alpha\chi = M^\alpha(s), \qquad d\chi/ds = Q(s) \tag{2.5}$$

The relationships (2.5) represent a system of three ordinary differential equations in the functions $\chi^\alpha$ and $\chi$ and are integrated explicitly; however, it will be more convenient later to deal with the differential form of writing (2.5). Because of (1.10), the $\chi^\alpha$ and $\chi$ determined from (2.5) are univalent functions on $\Gamma$.

The boundary conditions written in terms of the stress functions in the Kirchhoff model acquire the simple form

$$\nu_\alpha \frac{d\chi^\alpha}{ds} = M(s), \qquad \frac{d}{ds}\tau_\alpha \frac{d\chi^\alpha}{ds} = N(s) \tag{2.6}$$

The relationships (2.6) represent a system of equations in $\chi^\alpha$ and exactly as (2.5), are integrated explicitly. By using (1.11) it is easy to see that the corresponding solutions are univalent functions on the contour $\Gamma$.

The quantities $M^\alpha$ and $Q$ are related to $M, N$ and $Q$ by one-to-one relationships, hence the Kirchhoff forces $M$ and $N$ as well as the transverse force $Q$ can be considered given in the Reissner model. The Kirchhoff and Reissner boundary conditions in terms of the stress functions hence acquire a simple content: the Kirchhoff boundary conditions reduce to prescribing two functions $\chi^\alpha$ on $\Gamma$ and the Reissner boundary conditions reduce to prescribing the function $\chi \equiv {}^1/_2\chi_{,\alpha}^{\alpha}$ in addition to $\chi^\alpha$.

4°. Conditions on the discontinuity. It is sometimes convenient to use piecewise-smooth functions to construct the admissible fields $m^{\alpha\beta}$ and $q^\alpha$. The discontinuities in $m^{\alpha\beta}$, $q^\alpha$, $\chi^\alpha$ and $\chi$ cannot be arbitrary. Appropriate conditions on the discontinuities are derived from (1.7) and (1.8) and have identical form for the Kirchhoff and Reissner models

$$[\chi^\alpha] = cx^\alpha + c^\alpha, \quad [\chi] = c$$

Here $[A]$ is the difference in the values of the quantity $A$ on two sides of the line of discontinuity, and $c$, $c^\alpha$ are some constants.

5°. Variational principle for the Reissner model. The variational problem (1.1) — (1.4) is formulated as follows in terms of the stress functions: find the minimum of the functional

$$E = \int_\Omega \Phi_1 (\chi^\alpha_{,\beta}; \chi_{,\alpha})\, dx^1 dx^2 \tag{2.7}$$

$$\Phi_1 (\chi^\alpha_{,\beta}; \chi_{,\alpha}) = \Phi (e^{\beta\gamma}\chi^\alpha_{,\gamma} + e^{\alpha\beta}\chi; e^{\alpha\beta}\chi_{,\beta}), \qquad \chi = {}^1/_2\chi^\alpha_{,\alpha}$$

among all the functions $\chi^\alpha$ satisfying the conditions (2.5) on the contour $\Gamma$.

6°. Compatibility equations for the Reissner model. The Euler equations of the variational problem in the preceding item are

$$-e^{\beta\gamma}\left(\frac{\partial \Phi}{\partial m^{\alpha\beta}}\right)_{,\gamma} + \frac{1}{2}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \Phi}{\partial q^\gamma} e^{\gamma\beta}\right)_{,\beta} = 0 \tag{2.8}$$

After having substituted the expression (1.4) for $\Phi$ into (2.8), then (2.8) is converted into an equation in $m^{\alpha\beta}$ and $q^\alpha$ which closes the system of equilibrium equations (1.2) and (1.3).

In particular, in the isotropic case

$$2\Phi = 12\left[\frac{1}{2\mu} m_{\alpha\beta}m^{\alpha\beta} - \frac{\lambda}{2\mu\,(3\lambda + 2\mu)}(m_\gamma{}^\gamma)^2\right] + \frac{6h^2}{5\mu} q^\alpha q_\alpha$$

Hence, (2.8) becomes

$$\frac{\lambda}{3\lambda + 2\mu} m^\sigma_{\sigma,\gamma} e^{\alpha\gamma} - m^{\alpha\beta,\gamma} e_{\beta\gamma} + \frac{h^2}{10} q^{\gamma,\alpha\beta} e_{\gamma\beta} = 0 \tag{2.9}$$

It can be verified that substituting the equation of state

$$m_{\alpha\beta} = \frac{1}{12}\left[\frac{2\mu\lambda}{\lambda + 2\mu}\psi^\sigma_{,\sigma}\delta_{\alpha\beta} + 2\mu\psi_{(\alpha,\beta)}\right]$$

$$q_\alpha = \frac{5}{6} h^{-2}\mu\,(u_{,\alpha} + \psi_\alpha)$$

into (2.9) converts it into an identity.

**3. One estimate of the elastic energy.** In the neighborhood $\Omega'$ of the contour $\Gamma$ let it be possible to introduce a curvilinear $\rho$, $s$, coordinate system related to $x^\alpha$ by the formulas $x^\alpha = x_0{}^\alpha(s) - \rho\nu^\alpha(s)$. Here $x_0{}^\alpha(s)$ are functions giving the contour $\Gamma$ parametrically, $0 \leqslant s \leqslant L$, $0 \leqslant \rho \leqslant l$, $L$ is the length of $\Gamma$ and $l$ is a sufficiently small fixed number. The $\rho$, $s$ coordinate system is orthogonal, and the covariant components of the metric tensor are given by the formulas

$$g_{\rho\rho} = 1, \; g_{ss} = (1 + k\rho)^2, \; g_{\rho s} = 0, \; k = \nu_\alpha d\tau^\alpha / ds$$

Let us assume that the functions $x_0{}^\alpha(s)$ are triply differentiable, and the curvature $k(s)$ of the contour $\Gamma$ and its first derivative are bounded $|k(s)| \leqslant k_1$, $|dk/ds| \leqslant k_2$. Let $k_1 l < 1$ also, and let the parameter $h$ be so small that $h \leqslant l$, $hlk_2 \leqslant 1$.

Theorem. The inequality

$$E_0 \leqslant A \int_\Gamma (M^2 + L^2N^2 + hLQ^2)\, ds \tag{3.1}$$

where the constant $A$ depends only on $k_1$, $k_2$, $l$ and the elastic moduli holds for the elastic energy $E_0$ found by the Reissner model (*).

Lemma. We consider the problem of the minimum of the functional

$$I(\chi^\alpha) = \frac{1}{2} \int_\Omega (\chi_{\alpha,\beta}\chi^{\alpha,\beta} + h^2 \chi_{,\alpha}\chi^{,\alpha})\, dx^1 dx^2, \quad \chi = \frac{1}{2}\chi^\alpha_{,\alpha} \tag{3.2}$$

among the functions $\chi^\alpha$ satisfying the following conditions on $\Gamma$ :

$$\chi^\alpha |_\Gamma = \chi_0{}^\alpha, \quad \chi|_\Gamma = \chi_0$$

Then the estimate

$$I_0 \leqslant A \int_\Gamma \left[ \left( \nu_\alpha \frac{d\chi_0{}^\alpha}{ds} \right)^2 + h^2 \left( \frac{d}{ds} \tau_\alpha \frac{d\chi_0{}^\alpha}{ds} \right)^2 + \left( \tau_\alpha \frac{d\chi_0{}^\alpha}{ds} \right)^2 \right] ds \tag{3.3}$$

is valid for the minimal value $I_0$ of the functional $I$ .

Proof. Let us rewrite the integrand in (3.2) in the neighborhood $\Omega'$ of the contour $\Gamma$ in the curvilinear $\rho$, $s$ coordinate system by replacing the partial derivatives with respect to $x^\alpha$ by partial derivatives with respect to $\rho$, $s$ by the rules for differentiating a complex function, and by introducing the variables $\chi_\tau$ and $\chi_\nu$ in place of $\chi^\alpha$ by the relationships $\chi^\alpha = \chi_\nu \nu_\alpha + \chi_\tau \tau^\alpha$. The formulas

$$\rho_{,\alpha} = -\nu_\alpha(s), \qquad s_{,\alpha} = (1 + k\rho)^{-1}\tau_\alpha$$

$$\frac{d\tau^\alpha}{ds} = k\nu^\alpha, \quad \frac{d\nu^\alpha}{ds} = -k\tau^\alpha, \quad dx^1 dx^2 = |1 + k\rho|\, d\rho\, ds$$

are required here.

After simple estimates, we obtain (the comma in the subscripts denotes partial derivatives)

$$|1 + k\rho|\, \chi_{\alpha,\beta}\chi^{\alpha,\beta} \leqslant A\,[\chi^2_{\tau,s} + \chi^2_{\tau,\rho} + \chi^2_{\nu,s} + \chi^2_{\nu,\rho} + k_1^2(\chi_\tau^2 + \chi_\nu^2)] \tag{3.4}$$

$$|1 + k\rho|\, \chi_{,\alpha}\chi^{,\alpha} \leqslant A\,[\chi^2_{\tau,ss} + \chi^2_{\tau,\rho s} + \chi^2_{\nu,\rho s} + \chi^2_{\nu,\rho\rho} + (k_1^2 + L^2k_2^2)\chi^2_{\tau,s} + k_1^2(\chi^2_{\nu,s} + \chi^2_{\nu,\rho}) + (k_1^4 + k_2^4)\chi_\nu^2]$$

We specify the functions $\chi_\nu$ and $\chi_\tau$ in $\Omega'$ by the formulas

$$\chi_\tau = \chi_0{}^\alpha \tau_\alpha f(\rho), \qquad \chi_\nu = \chi_0{}^\alpha \nu_\alpha f(\rho) + \chi_0 g(\rho)$$
$$f(\rho) = (1 - \rho^2/l^2)^2, \qquad g(\rho) = \rho\,(1 - \rho^2/h^2)^2 \tag{3.5}$$

Outside of $\Omega'$ we set $\chi^\alpha = 0$. The functions constructed are admissible, hence their substitution into (3.2) yields the upper bound for $I_0$ . Because of this substitution, by using the inequalities

$$\int_\Gamma \chi_\alpha \chi^\alpha ds \leqslant A \int_\Gamma \left[ \left( \nu_\alpha \frac{d\chi^\alpha}{ds} \right)^2 + \left( \tau_\alpha \frac{d\chi^\alpha}{ds} \right)^2 \right] ds \tag{3.6}$$

$$\int_\Gamma \left[ \left( \frac{d}{ds} \nu_\alpha \chi^\alpha \right)^2 + \left( \frac{d}{ds} \tau_\alpha \chi^\alpha \right)^2 \right] ds \leqslant A \int_\Gamma \frac{d\chi_\alpha}{ds} \frac{d\chi^\alpha}{ds}\, ds$$

*) The letter $A$ will later denote constants independent of $k$.

$$\int\limits_{\Gamma}\left(\frac{d^2}{ds^2}\tau_\alpha\chi^\alpha\right)^2 ds \leqslant A\int\limits_{\Gamma}\left[\left(\frac{d}{ds}\tau_\alpha\frac{d\chi^\alpha}{ds}\right)^2 + L^{-2}\frac{d\chi^\alpha}{ds}\frac{d\chi_\alpha}{ds}\right]ds$$

as well as the inequalities (3.4), we obtain (3.3). The inequalities (3.6) are derived by using the Wirtinger inequalities [6].

Proof of the theorem. We can write for the elastic energy density

$$2\Phi \leqslant A\,(m_{\alpha\beta}m^{\alpha\beta} + h^2 q_\alpha q^\alpha) = A\,(\chi_{\alpha,\beta}\chi^{\alpha,\beta} - {}^1/_2\,(\chi^{\sigma}_{,\sigma})^2 + h^2\chi_{,\alpha}\chi^{,\alpha}) \leqslant A\,(\chi_{\alpha,\beta}\chi^{\alpha,\beta} + h^2\chi_{,\alpha}\chi^{,\alpha})$$

Hence, the minimum (2.7) of the functional $E$ among the functions satisfying the boundary conditions (2.5) will not exceed the minimum in the same set of functions of the functional $A\cdot I$. Hence, (3.1) results from the use of (2.6), (3.3) and the Wirtinger inequalities.

Note. It has not been used above that the tensor of the elastic compliances is independent of $x^\alpha$. Hence, the assertion proved is valid even for variable-thickness inhomogeneous plates.

**4. Asymptotic behavior of solutions of the Reissner equations.** Let us henceforth consider the external loads $M^\alpha$ and $Q$ independent of the parameter $h$, and $M^2 + N^2 \not\equiv 0$. Then $E_0$ evidently is on the order of $h^\circ = 1$.

Theorem. The solution of the system of Reissner equations converges, as $h \to 0$, to the solution of the Kirchhoff equations in energy form.

Proof. Let $m_K{}^{\alpha\beta}$, $q_K{}^\alpha$ and $m_R{}^{\alpha\beta}$, $q_R{}^\alpha$ be the solutions of the Kirchhoff and Reissner equations. $\Delta m^{\alpha\beta}$, $\Delta q^\alpha$ their difference, $E\,(\Delta m^{\alpha\beta}, \Delta q^\alpha)$ the elastic energy of the field $\Delta m^{\alpha\beta}$, $\Delta q^\alpha$. We show that

$$E\,(\Delta m^{\alpha\beta}, \Delta q^\alpha) = O\,(h) \tag{4.1}$$

The assertion of the theorem evidently follows from (4.1).

Let us consider the solution $m_K{}^{\alpha\beta}$ and $q_K{}^\alpha$ of the Kirchhoff equations known and let us replace the required functions in the variational problem (1.1)—(1.4) by $m'^{\alpha\beta} = m^{\alpha\beta} - m_K{}^{\alpha\beta}$, $q'^\alpha = q^\alpha - q_K{}^\alpha$. After discarding terms dependent only on $m_K^{\alpha\beta}$ and $q_K^\alpha$, the energy functional becomes

$$J = \int\limits_{\Omega}(\Phi\,(m'^{\alpha\beta}, q'^\alpha) + h^2 B_{\alpha\beta}q'^\alpha q_K{}^\beta + A_{\alpha\beta\gamma\delta}m'^{\alpha\beta}m_K^{\gamma\delta})\,dx^1\,dx^2 \tag{4.2}$$

The quantities $m'^{\alpha\beta}$ and $q'^\alpha$ are determined from the problem of the minimum of the functional $I$ for all functions $m'^{\alpha\beta}$ and $q'^\alpha$ satisfying (1.2) and the boundary conditions

$$\begin{aligned} m'^{\alpha\beta}\nu_\beta &= \tau^\alpha\,(M^\beta\tau_\beta - m_K^{\beta\gamma}\tau_\beta\nu_\gamma) \\ q'^\alpha\nu_\alpha &= -\frac{d}{ds}\,(M^\beta\tau_\beta - m_K^{\beta\gamma}\tau_\beta\nu_\gamma) \equiv Q' \end{aligned} \tag{4.3}$$

We note that the Kirchhoff boundary forces $M$ and $N$ are equal zero for the boundary conditions (4.3). The minimizing element of the function $J$ is $\Delta m^{\alpha\beta}$, $\Delta q^\alpha$.

Let us show that the last term in (4.2) vanishes. In fact, a function $u$ exists for the solution of the Kirchhoff equations such that $A_{\alpha\beta\gamma\delta}\;m_K{}^{\gamma\delta} = u_{,\alpha\beta}$, hence, the last term in (4.3) can be rewritten as

$$\int\limits_{\Omega} m'^{\alpha\beta}u_{,\alpha\beta}\,dx^1\,dx^2 \tag{4.4}$$

On the other hand, by virtue of (1.2) and (4.3), the integral (4.4) vanishes for any function $u$

Let us estimate the second term in (4.2) by using the Cauchy-Buniakowski inequalities

$$h^2\int_\Omega B_{\alpha\beta}q'^\alpha q_K{}^\beta\,dx^1\,dx^2 \leqslant \frac{h^2}{2}\int_\Omega\left(\frac{1}{2}B_{\alpha\beta}q'^\alpha q'^\beta + 2B_{\alpha\beta}q_K{}^\alpha q_K{}^\beta\right)dx^1\,dx^2 \leqslant \tag{4.5}$$
$$\frac{1}{2}\int_\Omega \Phi(m'^{\alpha\beta},\, q'^\alpha)\,dx^1\,dx^2 + h^2\int_\Omega B_{\alpha\beta}q_K{}^\alpha q_K{}^\beta\,dx^1\,dx^2$$

On the basis of (4.5), the upper and lower bounds of the functional $J$ are

$$\frac{1}{2}\int_\Omega \Phi(m'^{\alpha\beta},\, q'^\alpha)\,dx^1\,dx^2 - h^2\int_\Omega B_{\alpha\beta}q_K{}^\alpha q_K{}^\beta\,dx^1\,dx^2 \leqslant J \leqslant \tag{4.6}$$
$$\frac{3}{2}\int_\Omega \Phi(m'^{\alpha\beta},\, q'^\alpha)\,dx^1\,dx^2 + h^2\int_\Omega B_{\alpha\beta}q_K{}^\alpha q_K{}^\beta\,dx^1\,dx^2$$

Setting $m'^{\alpha\beta} = \Delta m^{\alpha\beta}$, $q'^\alpha = \Delta q^\alpha$, we have from the first inequality in (4.6)

$$\frac{1}{2}E(\Delta m^{\alpha\beta},\, \Delta q^\alpha) - h^2\int_\Omega B_{\alpha\beta}q_K{}^\alpha q_K{}^\beta\,dx^1\,dx^2 \leqslant \inf J \tag{4.7}$$

Minimizing both sides of the second inequality (4.6) in $m'^{\alpha\beta}$ and $q'^\alpha$

$$\inf I \leqslant \frac{3}{2}\inf\int_\Omega \Phi(m'^{\alpha\beta},\, q'^\alpha)\,dx^1\,dx^2 + h^2\int_\Omega B_{\alpha\beta}q_K{}^\alpha q_K{}^\beta\,dx^1\,dx^2 \tag{4.8}$$

We estimate the right side in (4.8) by using the inequality (3.1). Since $M = N = 0$ for the boundary conditions (4.3), we obtain

$$\inf\int_\Omega \Phi(m'^{\alpha\beta},\, q'^\alpha)\,dx^1\,dx^2 \leqslant Ah\int_\Gamma Q'^2\,ds \tag{4.9}$$

The assertion of the theorem and (4.1) follow from (4.7) and (4.9).

The question of convergence of the solutions of the Reissner equations to the solutions of the Kirchhoff equations was examined in [7] by another method.

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Translated by M. D. F.